10th Week

26 Nisan 2023 Çarşamba 14:27

$\mathbb{R}^3$, $\mathbb{R}^2$, $\mathbb{R}^{n\times n}$, $\mathbb{P}_n \to ax^{n-1} + bx^{n-2} - - - - + d$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\begin{bmatrix} a \\ b \end{bmatrix}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} - & - \\ - & - \end{bmatrix}_{n\times n}$

Span: Does $\{v_1, v_2, ..., v_t\}$ span V? $\rightsquigarrow$ $\text{Span}\{v_1, v_2, ..., v_t\} \stackrel{?}{=} V$

\# elements $= t$

$\rightsquigarrow$ Can we generate all vectors in V as a linear comb. of these vectors?

$r_1v_1 + r_2v_2 + ... + r_tv_t$ = a typical vector in V

system of linear equations

$\to$ if this system has a solution for $r_1, r_2, ..., r_t$ (without any restrictions) $\Rightarrow$ ✓

$\to$ if the system may not have a solution $\Rightarrow$ ✗

$t < \dim(V) \Rightarrow$ The set can not span V.

$t = \dim(V) \Rightarrow$ a square system $\det \neq 0 \Rightarrow$ ✓ $\det = 0 \Rightarrow$ ✗

$t > \dim(V) \Rightarrow$ may or may not span V $\rightsquigarrow$ solve the system.

Linear Independence: Is $\{v_1, v_2, ..., v_t\}$ linearly independent?

\# elements $= t$

$\vec{0}_V$ in the set $\Rightarrow$ ✗

if the vectors can be written as an linear comb. $\Rightarrow$ ✗

$c_1v_1 + c_2v_2 + ... + c_tv_t = 0_V$

a system of linear equations

$c_1 = c_2 = ... = c_t = 0 \to$ trivial soln. is the only soln. $\Rightarrow$ ✓

inf. many soln. case $\Rightarrow$ ✗

$t < \dim(V) \Rightarrow$ may or may not be lin. indep. $\rightsquigarrow$ solve the system.

$t = \dim(V) \Rightarrow$ square system $\det \neq 0 \Rightarrow$ ✓ $\det = 0 \Rightarrow$ ✗

$t > \dim(V) \Rightarrow$ the set can not be lin. independent ✗

BASIS: $\{v_1, v_2, ..., v_t\}$ span ✓ + lin. indep. ✓

$t = \dim(V)$, $\det \neq 0$

Dimension

$2v_1 + 3v_2 + ... + -5v_t$ = Halil

$-1v_1 + 4v_2 + ... + 0v_t$ = Cihan F.

standard bases

$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

standard bases

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\dim(\mathbb{R}^3) = 3$ $\quad \mathbb{R}^3 \quad$ $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{e_3} \right\} = E$

$\dim(\mathbb{R}^2) = 2$ $\quad \mathbb{R}^2 \quad$ $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{e_2} \right\} = E$

$\dim(\mathbb{R}^{2\times 2}) = 4$ $\quad \mathbb{R}^{2\times 2} \quad$ $\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{e_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{e_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{e_4} \right\} = E$

$\dim(\mathbb{P}_3) = 3$ $\quad \mathbb{P}_3 \quad$ $\{1, x, x^2\}$ $\quad ax^2 + bx + c$

Finding a Basis for Subspaces

any subspace dimension $\leqslant$ dimension of the superspace

Ex

All of the answers are correct.

Problem 2. ANSWERS ONLY CHECKED -- ANSWERS NOT RECORDED

(4 points) Let V be the vector space of symmetric 2×2 matrices and W be the subspace

$\dim(\mathbb{R}^{2\times 2}) = 4$

$\longrightarrow W = \operatorname{span}\{ \underbrace{\begin{bmatrix} -1 & -2 \\ -2 & -2 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}}_{v_2} \}. \leqslant \mathbb{R}^{2\times 2} \Rightarrow \boxed{\dim(W) = 2}$

Find a nonzero element X in W.

$X = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

(Gives credit only if all entries of the matrix are true)

$\longrightarrow W = \{ rv_1 + sv_2 : r, s \in \mathbb{R} \}$

$1v_1 + 0v_2$

Entered	Answer Preview	Correct	Result
$\begin{bmatrix} - & - \\ - & - \end{bmatrix}$		$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$	incorrect

This answer is NOT correct.

$\{v_1, v_2\}$ spans W → is this a basis W? ✓

$\{v_1, v_2\}$ is also lin. indep.

! if you're given 2 vector (whatever vector space you're in), the only possible linear combination → scalar multiple.

Ex

$S = \left\{ r\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$, Find a basis for S. $\dim(S) = ?$

(S is a subspace of $\mathbb{R}^3$)

$= \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v_2}, \underbrace{\cancel{\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}}}_{v_3} \right\}$

$\{v_1, v_2, v_3\}$ already spans S.

$v_3 = v_1 + v_2 \rightarrow$ we don't need v_3. $\{v_1, v_2, v_3\}$ is not linearly indep.

$\{v_1, v_2\}$ is a linearly independent. still spans S.

A basis for S $= \{v_1, v_2\}$

A basis for $S = \{v_1, v_2\}$

$\dim(S) = 2$

$\mathbb{R}^2 \not\subseteq \mathbb{R}^3$

Ex In $v_1 = 1 + x$ $\quad v_2 = x^2$ $\quad v_3 = 2x^2 - 3$ $\quad \in \mathbb{P}_4$ $\quad \mathbb{P}_2 \leq \mathbb{P}_3 \leq \mathbb{P}_4$

$S = \text{Span}\{v_1, v_2, v_3\}$ Find a basis for S. $\dim(S) = ?$

$\{v_1, v_2, v_3\}$ spans S. (*) Is $\{v_1, v_2, v_3\}$ lin. independent?

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 + 0x + 0x^2 + 0x^3 \rightarrow 0_{\mathbb{P}_4}$$

$$c_1(1+x) + c_2(x^2) + c_3(2x^2 - 3) = 0 + 0x + 0x^2 + 0x^3 \rightarrow 0_{\mathbb{P}_4}$$

$$\begin{aligned} c_1 - 3c_3 &= 0 \Rightarrow c_3 = 0 \\ c_1 &= 0 \\ c_2 + 2c_3 &= 0 \Rightarrow c_2 = 0 \\ 0 &= 0 \end{aligned}$$

$c_1 = c_2 = c_3 = 0$ is the only solution of the system.

$\Rightarrow \{v_1, v_2, v_3\}$ is a linearly independent set. (**)

(*) + (**) $\Rightarrow$ $\{v_1, v_2, v_3\}$ is a basis for S.

$\dim(S) = 3$

Change of Bases

$\mathbb{R}^3$ any vector in this space

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = E$$

$e_1 \quad e_2 \quad e_3$

we fix the order

$$F = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\} \rightarrow \text{is a ordered basis for } \mathbb{R}^3$$

$v_1 \quad v_2 \quad v_3$

span ✓ $\quad$ lin. indep. ✓

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 4 \end{vmatrix} = -2 \begin{vmatrix} -1 & 0 \\ 1 & 4 \end{vmatrix} + 0 + 0 = 8 \neq 0$$

-4

Ex $\text{Ayşegül} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix} \in \mathbb{R}^3 \rightarrow v$

$v = E[v]_E \quad v = [v]_E$

$\text{Ayşegül} = 5e_1 + 8e_2 + 3e_3$

$\text{Ayşegül} = r_1 v_1 + r_2 v_2 + r_3 v_3$

$[v]_F \rightarrow$ coordinates of Ayşegül with respect to the basis $F = \{v_1, v_2, v_3\}$

$$F = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$$

Assume that you want to find r_1, r_2, r_3

$F = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$

Assume that you want to find r_1, r_2, r_3

$$r_1 v_1 + r_2 v_2 + r_3 v_3 = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix} \Rightarrow r_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + r_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + r_3 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$Ax = b$

$$\left.\begin{aligned} r_1 - r_2 + 0r_3 &= 5 \\ 2r_1 + 0r_2 + 0r_3 &= 8 \\ 3r_1 + 1r_2 + 4r_3 &= 3 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 2 & 0 & 0 & 8 \\ 3 & 1 & 4 & 3 \end{array}\right] \Rightarrow \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 4 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}}_{[v]_F} = \underbrace{\begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}}_{v}$$

$B = \{w_1, w_2, w_3\}$

$\det \neq 0$

F^{-1} exists ✓

$$F [v]_F = v$$

$$\underbrace{\cancel{F^{-1} F}}_{I} [v]_F = F^{-1} v$$

B, C two ordered basis for V. We don't know $v \in V$

Knowing B, C, and $[v]_B \Rightarrow$ how can we find $[v]_C = ?$

$$\cancel{v} = B [v]_B = C [v]_C = \_\_\_\_\_\_$$

$$\Rightarrow \boxed{C^{-1} B} [v]_B = \underbrace{\cancel{C^{-1} C}}_{I} [v]_C = [v]_C \checkmark$$

This matrix is called as the transition matrix from B to C.

Ex We're in $\mathbb{R}^2$. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$B = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right\}$ $\quad C = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

$\det = 12 - 10 = 2 \neq 0$ ✓ $\quad \det C = -2 - 0 = -2 \neq 0$ ✓

Halil $= v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

B and C are two ordered basis in $\mathbb{R}^2$.

$v = B[v]_B = C[v]_C = \ldots$

$$[v]_B = B^{-1} v = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

$8 - 15$ $\quad -4 + 9$

$\rightarrow$ coordinates of Halil wrt the basis B.

Short way to find ...? : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$B = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ $\quad B^{-1} = \begin{bmatrix} 4/2 & -5/2 \\ -2/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix}$

$\det B = 2$

short way to find 2x2 inverses : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} -2/2 & 3/2 \end{bmatrix} \begin{bmatrix} -1 & 3/2 \end{bmatrix}$

$\det B = 2$

$-21+25$

$-7\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \to$ Halil ✓

$-14+20$

EX

$B = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\} \qquad [v]_B = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$\det = 12 - 10 = 2 \neq 0$ ✓ $\qquad \det C = -2 - 0 = -2 \neq 0$ ✓

a) Find the transition matrix from B to C.

$B\,[v]_B = C\,[v]_C$

✓ ✓ ✓ ?

$[v]_C = (C^{-1}B)\,[v]_B$

b) $[v]_C = ?$

a) $C = \begin{bmatrix} -1 & 5 \\ 0 & 2 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} 2/-2 & -5/-2 \\ 0/-2 & -1/-2 \end{bmatrix} = \begin{bmatrix} -1 & 5/2 \\ 0 & 1/2 \end{bmatrix}$

$\det C = -2$

$C^{-1}B = \begin{bmatrix} -1 & 5/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$

$0+1 \qquad 0+2$

the transition matrix from B to C.

$-7 \;\; +10$

b) $[v]_C \qquad (C^{-1}B)\,[v]_B = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \end{bmatrix} = [v]_C$

(to check : $11\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \to$ Halil)

$-11+15$

$0+6$

7. Given

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad S = \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}$$

find vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ so that S will be the transition matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

B ? $\qquad$ C ✓

→ from B to C ✓ $= C^{-1}B$ ✓

$C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \qquad C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$\underbrace{C\,C^{-1}}_{I}B = B$ $\quad C^{-1}B \to S$

$\begin{bmatrix} 1 & 2 \\ \end{bmatrix} \begin{bmatrix} 3 & 5 \\ \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ \end{bmatrix} \longrightarrow B = \left\{ \begin{bmatrix} 5 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 9 & 4 \end{bmatrix} \longrightarrow B = \left\{ \underset{w_1}{\begin{bmatrix} 5 \\ 9 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} \right\}$$

8. Given

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad S = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \rightarrow C^{-1}B$$

find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that S will be the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

B C → ?

$$B = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$$

det = 8 − 6 = 2

$$B^{-1} = \begin{bmatrix} 4/2 & -1/2 \\ -6/2 & 2/2 \end{bmatrix}$$

$C^{-1}B$ ✓ $\quad B$ ✓ $\quad C = ?$

$$B^{-1} = \begin{bmatrix} 2 & -1/2 \\ -3 & 1 \end{bmatrix}$$

Find B^{-1}.

$$C^{-1}\underbrace{B\,B^{-1}}_{I} = C^{-1} \Rightarrow (C^{-1})^{-1} = C$$

$$C^{-1} = \underbrace{C^{-1}B}_{S}\,B^{-1} = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1/2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$$

−2+1 $\quad$ 8−1 $\quad$ −1+1

$\det C^{-1} = 0 - (-1) = 1$

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} \Rightarrow C = \left\{ \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1 \\ 5 \end{bmatrix}}_{u_2} \right\}$$